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Collisions of pulses can lead to holes via front interaction in the cubic-quintic complex Ginzburg-Landau equation in an annular geometry

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We study the interaction of counterpropagating pulse solutions for two coupled complex cubic-quintic Ginzburg-Landau equations in an annular geometry. For small approach velocity we find as an outcome of such collisions several results including zigzag bound pulses, stationary bound states of 2π holes, zigzag 2π holes, stationary bound states of π holes, zigzag bound states of π holes, propagating 2π holes, and propagating π holes as the real part of the cubic cross coupling between the counterpropagating waves is increased. We characterize in detail the collisions giving rise to the three states involving π holes as an outcome.

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Over the last few years it has become clear that the cubic-quintic complex Ginzburg-Landau (CGL) equation is becoming more and more important [1–4] for the description of dissipative optical solitons [5–9]. In particular, in Ref. [2] (and references cited therein) it has been argued that coupled cubic-quintic CGL equations are relevant to model twin-core fiber lasers. The cubic-quintic CGL equation is known to arise as a prototype envelope equation [10] near the onset of a weakly inverted oscillatory instability [11–13]. Thual and Fauve showed [14] that this equation permits stable stationary localized solutions in contrast to the cubic-quintic GL equation with real coefficients. Following the pioneering paper by Thual and Fauve a number of groups studied various aspects of localized solutions of the cubic-quintic CGL equation [15–20] including analytic approximation schemes [17,18], "two-particle" solutions [20], and stable hole solutions [19].

Hole solutions, which have been of interest in nonlinear optics for dispersive systems for decades, also became of interest for systems with a substantial amount of dissipation when Nozaki and Bekki showed analytically that the cubic CGL equation has stable hole solutions [21]. However, it became clear fairly soon that the Nozaki-Bekki holes are structurally unstable [13,22], meaning that, for example, adding a small stabilizing quintic term to the cubic CGL equation leads to the disappearance of the hole solutions. In 1991 Sakaguchi reported the stable existence of two classes of hole solutions for the cubic-quintic CGL equation [19]. Quite recently [3] we have shown that additional classes of hole solutions including breathing holes stably exist for the cubic-quintic CGL equation.

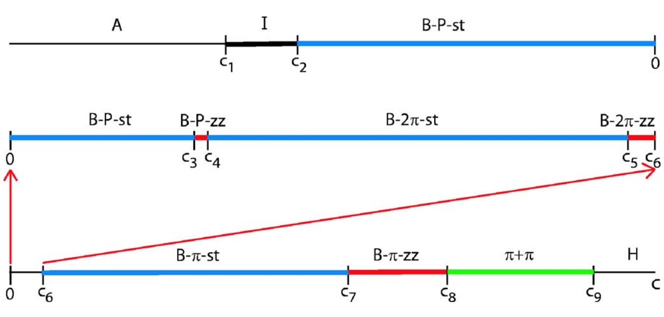

FIG. 1. (Color online) Results obtained for the interaction of two stationary pulses while keeping all parameters except for c_r and v fixed. A refers to annihilation, I to interpenetration, B-P-st to a stationary bound pair of pulses, B-P-zz to a zigzag bound pair of pulses, B-2π-st to a stationary bound pair of 2π holes, B-2π-zz to a zigzag bound pair of 2π holes, B-π-st to a stationary bound pair of π holes, B-π-zz to a zigzag bound pair of π holes, $\pi+\pi$ to counterpropagating π holes, and H to the spatially homogeneous solution. $\mu=-0.112$, $\beta_r=1$, $\beta_i=0.2$, $\gamma_r=-1$, $\gamma_i=0.15$, $D_r=1$, $D_i=0$, $c_r=v$, $c_i=0$, $c_1=-0.2$, $c_2=-0.166$, $c_3=0.001\,38$, $c_4=0.001\,46$, $c_5=0.0046$, $c_6=0.0048$, $c_7=0.047$, $c_8=0.061$, $c_9=0.081$.

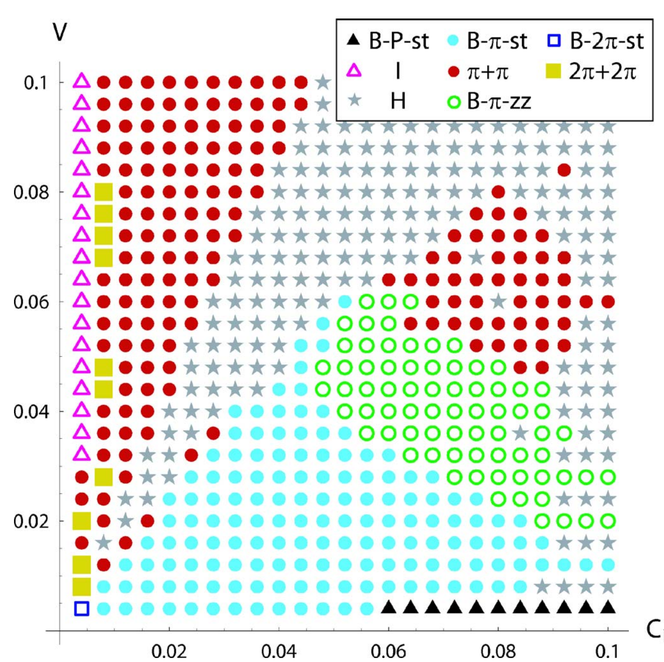

FIG. 2. (Color online) Phase diagram of possible outcomes resulting from the collision of two stationary pulses while keeping all parameters except for c_r and v fixed. I refers to interpenetration, B-P-st to a stationary bound pair of pulses, B-2π-st to a stationary bound pair of 2π holes, B-π-st to a stationary bound pair of π holes, B-π-zz to a zigzag bound pair of π holes, $\pi+\pi$ to counterpropagating π holes, $2\pi+2\pi$ to counterpropagating 2π holes, and H to the spatially homogeneous solution. The other parameters are as for Fig. 1.

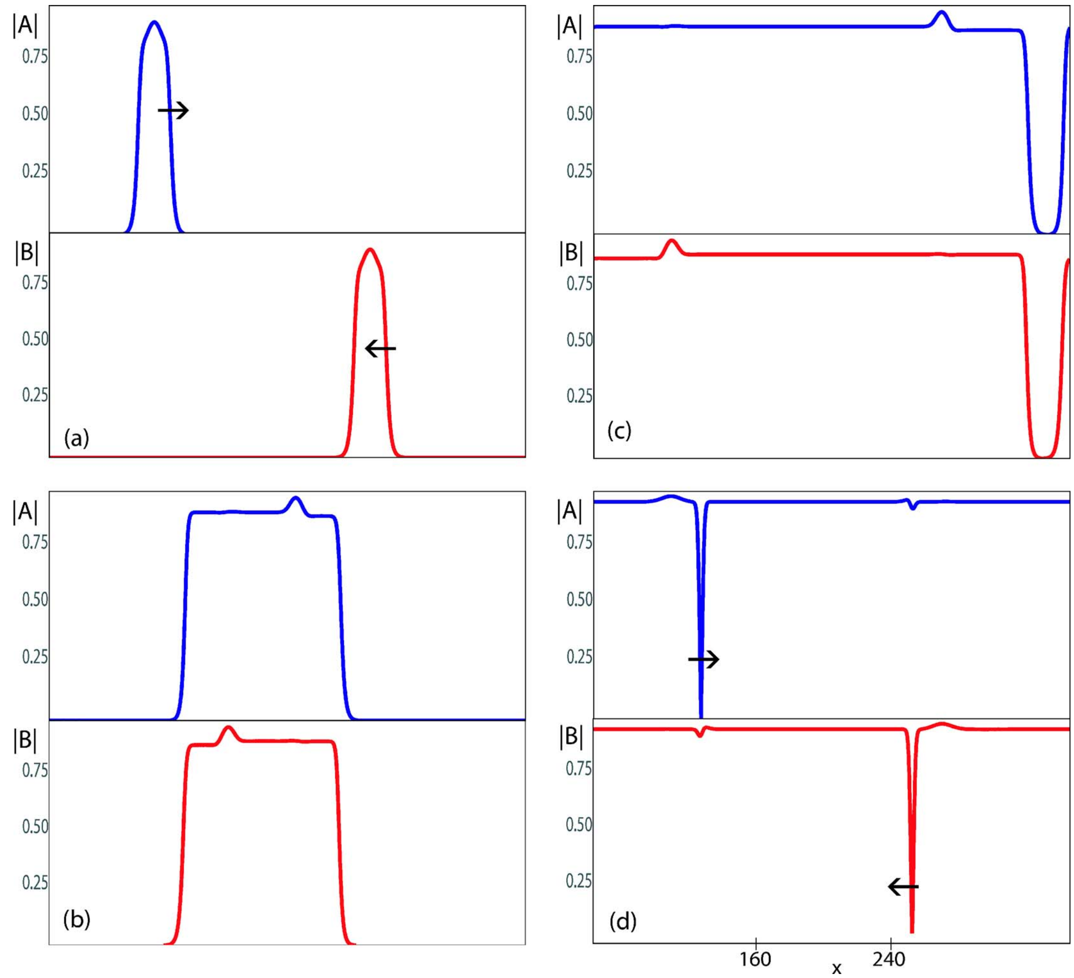

FIG. 3. (Color online) Time evolution of the interaction of two stationary pulse solutions resulting in two counterpropagating π holes for a box size $L=400$. The plots always show the moduli of the structures traveling to the right on top and the ones traveling to the left on the bottom for the four times $T=$(a) 0, (b) 1000, (c) 2600, and (d) 3800. $\mu=-0.112$, $\beta_r=1$, $\beta_i=0.2$, $\gamma_r=-1$, $\gamma_i=0.15$, $D_r=1$, $D_i=0$, $c_r=v=0.08$, $c_i=0$.

Much less work than on single localized solutions has been done for their interaction. For a single cubic-quintic CGL equation the interaction of two pulse solutions can lead to one pulse after the collision [15]; a similar result follows for hole solutions for which one type of hole was shown to survive after the interaction [19]. For coupled cubic-quintic CGL equations it has been shown that, as a function of the cross coupling between counterpropagating waves, interacting pulse solutions can annihilate, interpenetrate, form a stationary bound state of two pulses, or lead to a transition to the spatially homogeneous solution [20,23]. If a stationary pulse interacts with a pulse that is not in its asymptotic shape yet, it is also possible that one pulse survives after the collision [24]. For the interaction of breathing localized solutions rather complex behavior has been found, including a sensitive dependence on the initial phase of the breathing pulses [25]. For the cubic-quintic complex Swift-Hohenberg equation [26], a generalization of the evolution equation for a stationary instability derived for the order parameter by Swift and Hohenberg [27], similar results as for the interaction of stationary pulse solution for the cubic-quintic CGL equation have been found [26].

Here we show that as a function of the strength of the cubic cross coupling for counterpropagating waves and the approach velocity a number of very surprising phenomena not reported before arises when two pulse solutions interact in an annular geometry. These phenomena include bound states of 2π and π holes, which can either be stationary or show a zigzaglike spatiotemporal oscillation. In this connection we refer to a π hole, when there is a phase jump by π and when $|A|\equiv 0$ at the defect location. For 2π holes, $|A|$ does not touch zero near the center of the hole and the phase variation is continuous. Most importantly, however, it emerges that the interaction of two stationary counterpropagating pulse solutions can yield two counterpropagating π holes as a result of the interaction over a fairly large range of parameters.

The starting point of our investigations is two coupled subcritical cubic-quintic complex Ginzburg-Landau equations for counterpropagating waves with periodic boundary conditions:

$$\partial_t A - v\partial_x A = \mu A + (\beta_r + i\beta_i)|A|^2 A + (\gamma_r + i\gamma_i)|A|^4 A + (c_r + ic_i)|B|^2 A + (D_r + iD_i)\partial_{xx} A, \quad (1)$$

$$\partial_t B + v\partial_x B = \mu B + (\beta_r + i\beta_i)|B|^2 B + (\gamma_r + i\gamma_i)|B|^4 B + (c_r + ic_i)|A|^2 B + (D_r + iD_i)\partial_{xx} B, \quad (2)$$

where $A(x,t)$ and $B(x,t)$ are complex fields and where we have discarded quintic cross-coupling terms for simplicity. Equations (1) and (2) arise as prototype equations for a weakly inverted oscillatory bifurcation to traveling waves. A and B are slowly varying envelopes. The fast spatial and temporal variations have already been split off when writing down the coupled envelope equations. To compare with measurable quantities such as, for example, temperature variations in fluid dynamics, these rapid variations must be taken into account [1,10–12,28].

We have carried out a numerical study of Eqs. (1) and (2) using the following parameter values: $\mu=-0.112$, $\beta_r=-\gamma_r=D_r=1$, $\beta_i=0.2$, $\gamma_i=0.15$, $D_i=0$, and $c_i=0$. The coefficients β_r, D_r, and γ_r have been chosen so that stable pulse solutions are possible [14,16]: a stabilizing quintic term, a destabilizing cubic term, positive diffusion, and at least one nonvanishing imaginary part [29]. As a numerical method we used fourth-order Runge-Kutta finite differencing. We used typically $N=1000$ points with $dx=0.4$ (corresponding to a box length of $L=400$) and a time step of $dt=0.1$. For comparison we also studied smaller and larger box sizes to guarantee that there are no finite size effects for sufficiently large box size. We performed up to 2×10^6 iterations to check for long transients corresponding to a total time of $T=2\times 10^5$. We also changed dx and dt to verify that none of the results presented depends sensitively on the discretization used. We note that there is at maximum a small shift in the values of μ for which some of the solutions with a narrow range of stable existence arise. As initial conditions we used stationary pulses of a single cubic-quintic CGL equation in their asymptotic state: their envelope is not changing as a function of time and their speed is constant.

In Fig. 1 we have plotted our results along the line $v=c_r\equiv c$ while keeping all the other parameters fixed to the values given above. This choice has been made to cover a large fraction of the phenomena to be expected as both c_r and v, are varied. In Fig. 2 we have plotted the phase diagram for $0\leq v\leq 0.1$ and $0\leq c_r\leq 0.1$. Figure 2 shows that, while the diagram is rather complex in detail, almost all qualitatively different outcomes are already observed on the diagonal $v=c_r$. The complete, rather complex phase diagram in the vicinity of $c_r=v=0$ will be presented in [30].

Inspecting this "phase diagram," one realizes that, in addition to the known outcomes of collisions between two stationary pulses, namely, annihilation, interpenetration, a bound pair of stationary pulses, and the transition to the spatially homogeneous solution, there are six other classes of behavior. As c is increased above the interval for which stationary bound pulses are stable, one finds first between $c_3=0.001\,38$ and $c_4=0.001\,46$ a bound state of pulses undergoing a zigzag motion in time and space. Between $c_4=0.001\,46$ and $c_5=0.0046$ we find a bound state of stationary 2π holes, which is followed by a bound state of 2π holes undergoing a zigzag motion in time and space over the fairly narrow interval $c_5=0.0046$ and $c_6=0.0048$. Since the bound states are undergoing a zigzag motion in time and space, they are rather delicate objects and their basin of attraction is correspondingly fairly small. As c is increased further, we obtain a bound state of stationary π holes over the rather large range from $c_6=0.0048$ to $c_7=0.047$ followed by a bound state of π holes, which undergoes a zigzag motion in time and space over the range from $c_7=0.047$ to $c_8=0.061$. We note that the range of stable existence of the bound state of π holes showing a zigzag motion is much larger than that of the other two classes of bound states discussed above

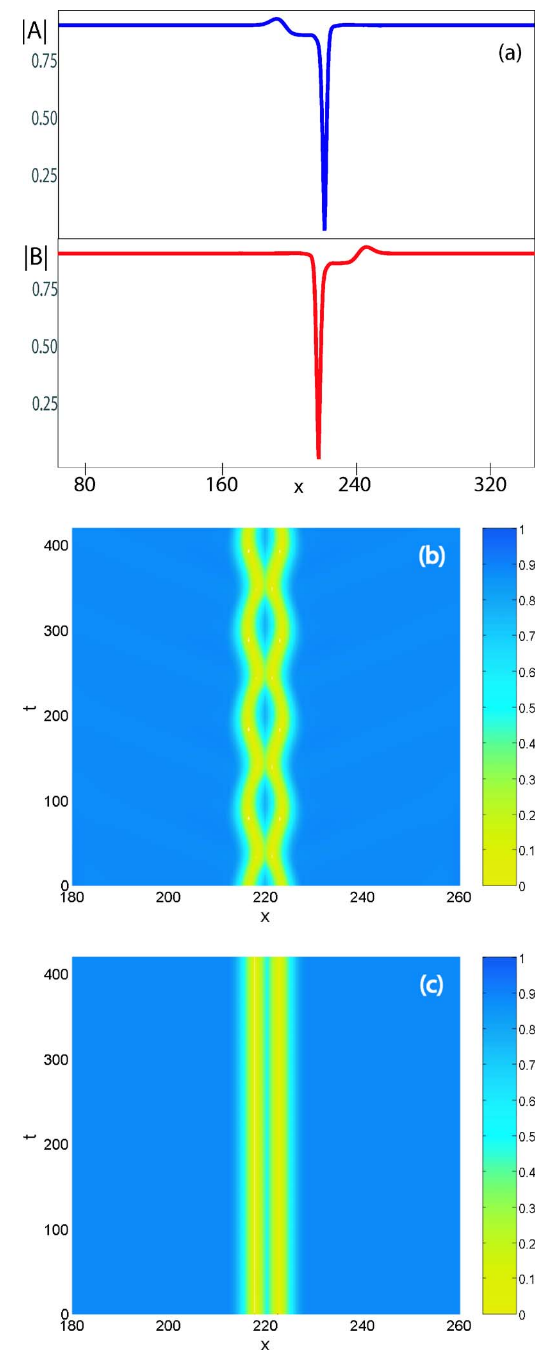

FIG. 4. (Color online) (a) Snapshot of the zigzag bound state of two π holes ($c_r=v=0.054$). (b) x-t plot for $\min(|A|,|B|)$ for the zigzag bound state of two π holes ($c_r=v=0.054$). (c) x-t plot for $\min(|A|,|B|)$ for a stationary bound state of two π holes ($c_r=v=0.04$). All other parameter values are as for Fig. 3.

showing a zigzag motion. Stable counterpropagating π holes exist between $c_8=0.061$ and $c_9=0.081$, when they are replaced by the spatially homogeneous solutions. It is the last three classes of states involving π holes that we will discuss in more detail in the following.

In Fig. 3 we have plotted the time evolution of the interaction between two stationary pulses leading to two propagating π holes as a result of the interaction. We have chosen $c_r=v=0.08$. We note that the localized region spreads as soon as the stationary pulse solutions start to interact, while maintaining a hump, which is a sink of traveling waves. Eventually two counterpropagating π holes result, which interact only very little, visible only via the small depression in the constant background of the hole propagating in the opposite direction.

As the magnitude of the cross coupling between counterpropagating waves, c_r, is reduced, one obtains a bound pair of π holes undergoing a spatiotemporal zigzag motion. This result is displayed in Fig. 4. In Fig. 4(a) we show a snapshot of this zigzag bound state at $c_r=v=0.054$ and in Fig. 4(b) a space-time plot of this state revealing a characteristic frequency of oscillation of the zigzag motion. As the value of the cubic cross-coupling term is further reduced a stationary bound state of two π holes results as is demonstrated in the x-t plot shown in Fig. 4(c) for $c_r=v=0.04$. The underlying mechanism for the transition to the zigzag bound state of two π holes is a Hopf bifurcation. We have thus shown that varying the magnitude and sign of the cubic cross-coupling coefficient c_r leads to a renormalization of μ and thus to many different states as a result. That this renormalization plays an important rule in the investigation of coupled cubic-quintic CGL equations was noticed first in Ref. [31], where the influence of noise was investigated.

In conclusion, we have shown in this Rapid Communication that, as the result of the interaction between two stationary propagating pulses, one obtains for coupled cubic-quintic complex CGL equations in an annular geometry a number of unexpected results. In addition to the results found in previous investigations we find five types of bound states: stationary bound states of 2π holes, and of π holes as well as bound states of pulses, 2π holes, and π holes which show a zigzag motion in space and time. The most important result of the present study, however, was the conversion of two counterpropagating stationary pulses into two counterpropagating π holes over a fairly large range in parameter space. Clearly this conversion deserves further detailed physical and mathematical investigations. Since the cubic-quintic CGL equation is a prototype equation, which is of growing importance, for example, for the field of nonlinear data transfer in optics, we anticipate that the results presented here will also stimulate experiments in the near future.

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